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RESONANCES IN MULTIFREQUENCY OSCILLATIONS

A. M. MOLČANOV

It is shown that resonance of general position is reduced by change of variables, by the introduction of resonance phases, to the special case of a single-frequency resonance when the frequency passes through zero. The concepts introduced are illustrated by the example of resonances in the solar system.

1. We shall study multifrequency oscillations of the form

$$\begin{aligned} \frac{dI}{dt} &= \varepsilon F(I, \varphi, \varepsilon), \\ \frac{d\varphi}{dt} &= \omega(I) + \varepsilon\Omega(I, \varphi, \varepsilon). \end{aligned} \tag{1}$$

Here ϵ is a small parameter; $I = (I_1, \cdots, I_k)$ are slowly varying variables, and $\phi = (\phi_1, \cdots, \phi_l)$ are rapidly varying (phase variables). The right-hand sides are smooth functions of their arguments and periodic with the period 2π for each phase.

The separation of the variables into rapidly and slowly varying variables has an asymptotic sense (as $\epsilon \to 0$). If we set $\epsilon = 0$, the slowly varying variables become the first integrals of the system $I = I_0$. The rapidly varying variables remain, in general, variables even when $\epsilon = 0$. However, some combinations of phases (we shall call them resonance phases) can become constants when $\epsilon = 0$. For a more precise description of the situation we shall introduce some definitions.

Definition 1. The surface in space I of slowly varying variables defined by the equality

$$(\mathbf{n}, \vec{\omega}) \equiv n_1\omega_1(I) + \ldots + n_l\omega_l(I) = 0, \tag{2}$$

where $\mathbf{n} = (n_1, \cdots, n_l)$ is an integral vector is called a *resonance surface* and $\mathbf{n}$ is a resonance vector.

Definition 2. An integral linear combination of phases

$$\psi = n_1\varphi_1 + \ldots + n_l\varphi_l \tag{3}$$

is called a *resonance phase* of a given resonance.

Definition 3. The number s of linearly independent resonance relations which are satisfied by point I is called the *index of complexity of point* I (or the *complexity of state* I).

2. In space I of slowly varying variables, the overwhelming majority of points, a set of full measure, are points with the index zero not lying on any of the resonance surfaces (2). If we make use of the terminology of probability theory, as is often done, we can say that the "probability of a point having the index 1 is equal to zero." It is essential, however, to take into consideration the slow motion, whose one-dimensional trajectories must intersect resonance surfaces of index 1. Therefore, from the standpoint of measure theory [1], it seems well to study resonances of index 1. As for resonances of complexity 2, the "probability" of encountering such a point in a practical problem is

equal to zero.

The result of analyzing the resonance relations that take place in the solar system is all the more unexpected and remarkable. Jumping ahead, let us here stress the main point. *In all cases analyzed, the complexity has the maximum possible value.*

It is clear from the foregoing that this fact cannot be explained by randomicity. Another possible explanation is significantly more satisfactory, namely that the total resonance of all subsystems in the solar system is the inevitable result of long evolution.

3. **The canonical form of a system of resonances.** Henceforth, when studying phenomena taking place in passing through a resonance, it is very important to be able to introduce precisely the resonance phases of the given system of resonances as independent phase variables. The change of phase variables

$$\psi = A\varphi, \qquad \varphi = B\psi \tag{4}$$

should preserve the form of system (1), that is, its periodicity in respect to phases. This means that both matrices A and B should be integral.

The following theorem plays an important role in this investigation.

Theorem of biorthogonalization. *If the integral vectors* $\mathbf{n}_1, \cdots, \mathbf{n}_s$ *are linearly independent, then there exists an integral biorthogonal system* $\mathbf{a}_1, \cdots, \mathbf{a}_s$; $\mathbf{b}_1, \cdots, \mathbf{b}_s$ *such that the vectors* $\mathbf{n}_1, \cdots, \mathbf{n}_s$ *are obtained from the vectors* $\mathbf{a}_1, \cdots, \mathbf{a}_s$ *by the integral triangular transformation*

$$\begin{aligned} \mathbf{n}_1 &= T_{11}\mathbf{a}_1, \\ \mathbf{n}_2 &= T_{21}\mathbf{a}_1 + T_{22}\mathbf{a}_2, \\ &\cdots\cdots\cdots \\ \mathbf{n}_s &= T_{s1}\mathbf{a}_1 + T_{s2}\mathbf{a}_2 + \ldots + T_{ss}\mathbf{a}_s. \end{aligned} \tag{5}$$

The theorem is proved by a method analogous to the well-known Lagrange method of orthogonalization [2], but with modifications due to the integral nature of the problem. The main modification is the replacement of an orthonormal system by a biorthonormal one [2]. A lemma which readily follows from the Euclidean algorithm is used in the proof [3].

Lemma. *For any integral vector* $\mathbf{a}$ *we find an integral vector* $\mathbf{b}$ *such that their scalar product* $d = (\mathbf{a}, \mathbf{b})$ *is equal to the largest common divisor of the components of vector* $\mathbf{a}$.

The theorem of the reduction of an arbitrary system of resonances

$$(\mathbf{n}_1, \vec{\omega}) = 0, \ldots, (\mathbf{n}_s, \vec{\omega}) = 0 \tag{6}$$

to the canonical form

$$\omega_1(I) = 0, \ldots, \omega_s(I) = 0 \tag{7}$$

follows almost directly from the theorem formulated above. For the proof, we must supplement the resonance vectors $\mathbf{n}_1, \cdots, \mathbf{n}_s$ in an arbitrary manner up to the integral basis $\mathbf{n}_1, \cdots, \mathbf{n}_s$, $\mathbf{n}_{s+1}, \cdots, \mathbf{n}_l$ and biorthogonalize this basis. The vectors $\mathbf{a}_i$, arranged in rows form the integral square matrix A and the columns $\mathbf{b}_k$ the matrix B, biorthogonality meaning simply that

$$AB = E. \tag{8}$$

Therefore matrices A and B can be used to construct the replacement of variables in (4). In this case, the first s phases will automatically be resonance phases due to the triangularity of the transformation (5).

4. **Resonances and the solar system.** From the standpoint of oscillation theory, any planetary system is a set of weakly coupled systems, the number of phases being equal to the number of planets. The solar system contains at least four such subsystems, namely the nine planets, four satellites of Jupiter, eight satellites of Saturn, and the five satellites of Uranus.

Since resonance relations correspond to the undisturbed problem ($\epsilon = 0$), in computations involving a practical problem it is necessary to consider the vector $\mathbf{n}$ as a resonance vector if its scalar product with the frequency vector $\vec{\omega}$, even though not zero, is of the order of ϵ: $(\mathbf{n}, \vec{\omega}) \sim \epsilon$. For the solar system, where $\epsilon \sim 10^{-3}$ (the ratio of the mass of the planets to the mass of the sun), this yields a value of the scalar product of several thousandths. The following expectation is justified: the principal resonance of the solar system is the resonance of 5:2 for the frequencies of Jupiter and Saturn; it has an accuracy of about ⅔% (0.0067). Of the 22 resonances cited below, only three are less accurate, even the very "worst" of them, the 1:2 resonance for Neptune and Uranus, nevertheless has an accuracy of 1.5%. In the table of frequencies [4], the frequency of the most massive body in each system is taken as unity.

Table of Frequencies

Planets

Mercury	49.22
Venus	19.29
Earth	11.862
Mars	6.306
Jupiter	1.000
Saturn	0.4027
Uranus	0.14119
Neptune	0.07197
Pluto	0.04750

Satellites of Jupiter

Io	4.044
Europa	2.015
Ganymede	1.000
Callistro	0.4288

Satellites of Uranus

Miranda	6.529
Ariel	3.454
Umbriel	2.100
Titania	1.000
Oberon	0.6466

Satellites of Saturn

Mimas	16.918
Enceladus	11.639
Tethys	8.448
Dione	5.826
Rhea	3.530
Titan	1.000
Hyperion	0.7494
Iapetus	0.2010

Table of Resonance Vectors

Planets

$$
\begin{array}{l}
(\;1\;\;\;1\;\;\;2\;\;1\;\;\;0\;\;0\;\;\;0\;\;0\;\;0)\\
(-0\;-1\;\;\;0\;\;3\;\;\;0\;\;1\;\;\;0\;\;0\;\;0)\\
(\;0\;\;\;0\;-1\;\;2\;-1\;\;1\;-1\;\;0\;\;0)\\
(\;0\;\;\;0\;\;\;0\;\;1\;-6\;\;0\;-2\;\;0\;\;0)\\
(\;0\;\;\;0\;\;\;0\;\;0\;-2\;\;5\;\;\;0\;\;0\;\;0)\\
(\;0\;\;\;0\;\;\;0\;\;0\;-1\;\;0\;\;\;7\;\;0\;\;0)\\
(\;0\;\;\;0\;\;\;0\;\;0\;\;\;0\;\;0\;-1\;\;2\;\;0)\\
(\;0\;\;\;0\;\;\;0\;\;0\;\;\;0\;\;0\;-1\;\;0\;\;3)
\end{array}
$$

Satellites of Saturn

$$
\begin{array}{l}
(1\;\;\;0\;-2\;\;\;0\;\;0\;\;\;0\;\;\;0\;\;\;0)\\
(0\;-1\;\;\;0\;\;\;2\;\;0\;\;\;0\;\;\;0\;\;\;0)\\
(0\;\;\;0\;-1\;\;\;0\;\;2\;\;\;1\;\;\;0\;\;\;2)\\
(0\;\;\;0\;\;\;0\;-1\;\;2\;-1\;\;\;0\;-1)\\
(0\;\;\;0\;\;\;0\;\;\;0\;\;1\;-2\;-2\;\;\;0)\\
(0\;\;\;0\;\;\;0\;\;\;0\;\;0\;\;\;3\;-4\;\;\;0)\\
(0\;\;\;0\;\;\;0\;\;\;0\;\;0\;-1\;\;\;0\;\;\;5)
\end{array}
$$

Satellites of Jupiter

$$
\begin{array}{l}
(1\;-2\;\;\;0\;\;0)\\
(0\;\;\;1\;-2\;\;0)\\
(0\;-3\;\;\;0\;\;7)
\end{array}
$$

Satellites of Uranus

$$
\begin{array}{l}
(-1\;\;1\;\;\;1\;\;\;1\;\;0)\\
(\;\;0\;\;1\;-1\;-2\;\;1)\\
(\;\;0\;\;0\;-2\;\;\;1\;\;5)\\
(\;\;0\;\;0\;\;\;1\;-4\;\;3)
\end{array}
$$

An analysis of the tables of resonances leads to the following conclusions:

1. The rule of maximum resonance is applicable to all systems of satellites and to the system of planets, namely the number of resonance relations is equal to one less than the number of phases.

2. Systems with a small number of members are quite homogeneous; the resonance relations include the majority of participants.

3. The system of planets and the satellites of Saturn (consisting of 9 and 8 members, respectively) show a clear tendency to creating a heterogeneous structure.

Indeed, the system of planets naturally divides into three types: Mercury–Venus–Earth–Mars, Jupiter–Saturn, and Uranus–Neptune–Pluto. There are unifying resonances within each group: the first, second, and third for the Earth group, the fifth for the Jupiter group, and the seventh and eighth for the Uranus group, the subordinate role of the group of Earth-like planets relative to the other groups being quite obvious. The fourth and sixth resonances which unify three coalitions into a single solar system play the principal role.

The system of satellites of Saturn exhibits a similar structure, but the coalitions are more nearly equal. The coalitions are as follows: Mimas–Tethys, Enceladus–Dione, Rhea–Titan–Hyperion–Iapetus. It is easy to find "local" and "general" resonances in the table.

5. The law of planetary distances. We shall note an important property of states with a maximum index of complexity. States with a maximum index of complexity are given uniquely by tables of resonances. Indeed, in this case, all frequencies can be expressed through one frequency which remains free. A change in the free frequency simply corresponds to a change in the scale of the system.

For planetary systems, this really means that the formulation of the question of the law of planetary distances [5] is unfortunate since simple integral relations are obtained not for the distances but for the frequencies through which the distances are uniquely determined. Moreover, the law of planetary distances, if it is applicable, is only for planets, while the rule of maximum complexity of resonance is equally applicable to all analyzed cases.

It is impossible to refrain from commenting that the integrality which is usually associated with quantum physics is probably the common property of sufficiently old systems. It is simply that quantum systems are always old for us since their time scale is usually negligible and we "find" them already passed through their evolution. It is possible that this is precisely why integrality has drawn attention to itself first of all in the physics of elementary particles and has even determined the name quantum.

The basic idea can be formulated briefly as follows: the resonance of a system is a consequence (and criterion) of its evolutionary maturity.

This viewpoint finds a curious substantiation in the fact usually noted as an amusing happenstance. It is known [6] that motion in an arbitrary centrally symmetric field has, generally speaking, two frequencies, the angular and radial frequencies not being connected with each other in any way. However, in the case of a Newtonian potential, we do have an identical (for all values of momentum and energy) resonance 1:1 of these frequencies. Consequently, the motion of a single planet satisfies the rule of maximum resonance. This fact is very satisfactory from an evolutionary standpoint since the Newtonian potential corresponds to interactions with the largest time scales, the electrical and gravitational.

Still another resonance potential is known, namely the potential of a harmonic oscillator with a 1:2 resonance. It would be interesting to prove that other identical resonance potentials do not exist.

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Translated by:
E. J. Osten